Coordinate Singularities

This kind of singularity is an apparent problem brought on by the particular choice of coordinates. For example, consider the line element ds on the surface of a sphere of radius R, expressed in spherical coordinates.

$$ds^2 = R^2 d\phi^2 + R^2(\sin\phi)^2 d\theta^2,$$

where ϕ is the polar angle, and θ is the azimuthal angle. We see that for $\phi = 0$, there are an infinite number of values of θ that correspond to the same point (the "North Pole"). This seems to indicate some sort of problem.

But there is nothing odd about the top point on the sphere; we could transform the coordinates so that some point on the equator corresponded to zero polar angle. So the apparent problem is caused by the choice of coordinates.

For a more serious example of a coordinate singularity, consider the line element in the flat, two-dimensional plane, expressed in polar coordinates:

$$ds^2 = dr^2 + r^2 d\phi^2.$$

If we make the transformation: $r = \frac{a^2}{r'^2}$, where a is some constant, the line element becomes:

$$ds^2 = \frac{a^4}{r'^4}(dr'^2 + r'^2 d\phi^2),$$

and there now seems to be a serious problem at r' = 0. Nevertheless, there are no real singularities in the flat plane. The blowing up of the line element as r' approaches zero is an artifact caused by the choice of coordinates.

Now let us consider the Schwarzschild line element. We know that the coefficient of dr^2 for this metric is $\left(1 - \frac{2M}{r}\right)^{-1}$, which blows up at r = 2M. To show that this is only an apparent singularity caused by the choice of coordinates, transform to another coordinate system for which the singularity disappears. One choice is Eddington-Finkelstein coordinates, in which the time coordinate is replaced by another coordinate labeled v, with the transformation equation:

$$t = v - r - 2M ln\left|\frac{r}{2M} - 1\right|.$$

With this transformation, the Schwarzschild metric becomes:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2vdr + r^2(d\phi^2 + (\sin\phi)^2 d\theta^2).$$

Now we see that, although the coefficient of dv^2 goes to zero, which is a bit strange, nothing blows up to infinity. Physically, the radius $r = 2M$ is not a singularity, but significant physical events happen there, such as the changing of signs of the time and space coefficients, which makes the radius act like a time coordinate, always moving in one direction (towards $r = 0$).

On the other hand, the point $r = 0$ is what physicists call an essential singularity, and there is no transformation of coordinates for which the point $r = 0$ will not cause the line element to have an infinite coefficient.